Random hyperbolic graphs in $d+1$ dimensions

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We generalize random hyperbolic graphs to arbitrary dimensionality. We find the rescaling of network parameters that allows to reduce random hyperbolic graphs of arbitrary dimensionality to a single mathematical framework. Our results indicate that RHGs exhibit similar topological properties, regardless of the dimensionality of their latent hyperbolic spaces.

I. INTRODUCTION

Random hyperbolic graphs (RHGs) have been introduced as latent space models, where nodes correspond to points in the 2-dimensional hyperboloid $\mathbb{H}^2$ and connections between the nodes are established with distance-dependent probabilities [1, 2]. RHGs have been shown to accurately model structural properties of real networks including sparsity, self-similarity, scale-free degree distribution, strong clustering, the small-world property, and community structure [2–6]. Using the RHG as a null model, one can map real networks to hyperbolic spaces [7–9]. Practical interest in the RHGs is driven by promising applications in routing and navigation [7, 10–13], link prediction [8, 14–20], network scaling [3, 6, 12, 21], and semantic analysis [22].

Of relevance to this work and also of potential interest to the reader is the hyperbolic graph generator [23], allowing to generate RHGs in $\mathbb{H}^2$, as well as the analysis of small-world and clustering in related $\mathbb{S}^d$ models, defined in *Euclidean* spaces [24].

In this work we extend the RHG to the arbitrary dimensionality, compute its basic structural properties, and discuss its limiting regimes.

II. HYPERBOLIC RANDOM GRAPH MODEL IN $d+1$ DIMENSIONS

Consider the upper sheet of the $d+1$-dimensional hyperboloid of curvature $K=-\zeta^2$

$$x_0^2 - x_1^2 - \ldots - x_{d+1}^2 = \frac{1}{\zeta^2},\ x_0 > 0 \tag{1}$$

in the $d+2$-dimensional Minkowski space with metric

$$\mathrm{d}s^2 = -\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \ldots + \mathrm{d}x_{d+1}^2. \tag{2}$$

The spherical coordinate system on the hyperboloid $(r, \theta_1, \ldots, \theta_d)$ is defined by

$$\begin{aligned}
x_0 &= \frac{1}{\zeta}\cosh\zeta r,\\
x_1 &= \frac{1}{\zeta}\sinh\zeta r\cos\theta_1,\\
x_2 &= \frac{1}{\zeta}\sinh\zeta r\sin\theta_1\cos\theta_2,\\
&..\\
x_d &= \frac{1}{\zeta}\sinh\zeta r\sin\theta_1\ldots\sin\theta_{d-1}\cos\theta_d,\\
x_{d+1} &= \frac{1}{\zeta}\sinh\zeta r\sin\theta_1\ldots\sin\theta_{d-1}\sin\theta_d,
\end{aligned} \tag{3}$$

where $r>0$ is the radial coordinate and $(\theta_1, \ldots, \theta_d)$ are the standard angular coordinates on the unit d-dimensional sphere $\mathbb{S}^d$.

The coordinate transformation in (3) yields the $\mathbb{H}^{d+1}$ metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \frac{1}{\zeta^2}\sinh^2(\zeta r)\,\mathrm{d}\Omega_d^2, \tag{4}$$

$$\mathrm{d}\Omega_d^2 = \mathrm{d}\theta_1^2 + \sin^2(\theta_1)\mathrm{d}\theta_2^2 + .. + \sin^2(\theta_1)\ldots\sin^2(\theta_{d-1})\mathrm{d}\theta_d^2, \tag{5}$$

resulting in the volume element in $\mathbb{H}^{d+1}$:

$$\mathrm{d}V = \left(\frac{1}{\zeta}\sinh\zeta r\right)^d \mathrm{d}r\prod_{k=1}^{d}\sin^{d-k}(\theta_k)\mathrm{d}\theta_k. \tag{6}$$

The distance between two points i and j in $\mathbb{H}^{d+1}$ is given by the hyperbolic law of cosines:

$$\cosh\zeta d_{ij} = \cosh\zeta r_i\cosh\zeta r_j - \sinh\zeta r_i\sinh\zeta r_j\cos\Delta\Theta_{ij}, \tag{7}$$

where $\Delta\Theta_{ij}$ is the angle between i and j:

$$\cos(\Delta\Theta_{ij}) = \cos\theta_{i,1}\cos\theta_{j,1} + \sin\theta_{i,1}\sin\theta_{j,1}\cos\theta_{i,2}\cos\theta_{j,2} + ... + \sin\theta_{i,1}\sin\theta_{j,1}...\sin\theta_{i,d-1}\sin\theta_{j,d-1}\cos\theta_{i,d}\cos\theta_{j,d} + \sin\theta_{i,1}\sin\theta_{j,1}...\sin\theta_{i,d-1}\sin\theta_{j,d-1}\sin\theta_{i,d}\sin\theta_{j,d}, \quad (8)$$

$(\theta_{i,1}, ..., \theta_{i,d})$ and $(\theta_{j,1}, ..., \theta_{j,d})$ are the coordinates of points i and j on $\mathbb{S}^d$.

For sufficiently large ζr_i and ζr_j values, the hyperbolic law of cosines in Eq. (7) is closely approximated by

$$d_{ij} = r_i + r_j + \frac{2}{\zeta}\ln\left(\sin(\Delta\Theta_{ij}/2)\right). \quad (9)$$

The hyperbolic ball $\mathbb{B}^{d+1}$ of radius $R_H > 0$ is defined as the set of points with

$$r \in [0, R_H]. \quad (10)$$

Nodes of the HRG are points in $\mathbb{B}^{d+1}$ selected at random with density $\rho(\mathbf{x}) \equiv \rho(r)\rho(\theta_1)...\rho(\theta_d)$, where

$$\begin{aligned}
&\rho(r) = \left[\sinh(\alpha r)\right]^d / C_d, \quad \alpha \geq \zeta/2\\
&C_d \equiv \int_0^{R_H} \left[\sinh(\alpha r)\right]^d \mathrm{d}r,\\
&\rho_k(\theta) = \left[\sin(\theta)\right]^{d-k} / I_{d,k},\\
&I_{d,k} \equiv \int_0^{\pi} \left[\sin(\theta)\right]^{d-k} \mathrm{d}\theta = \sqrt{\pi}\frac{\Gamma\left[\frac{d-k+1}{2}\right]}{\Gamma\left[1+\frac{d-k}{2}\right]} \quad k \in [1, d-1],\\
&I_{d,d} \equiv 2\pi,
\end{aligned} \quad (11)$$

In other words, nodes are uniformly distributed on unit sphere $\mathbb{S}^d$ with respect to their angular coordinates. In the special case of $\alpha = \zeta$ nodes are also uniformly distributed in $\mathbb{B}^{d+1}$.

Pairs of nodes i and j are connected independently with connection probability

$$p_{ij} = p\left(d_{ij}\right) = \frac{1}{1 + \exp\left(\zeta\left(d_{ij} - \mu\right)/2T\right)}, \quad (12)$$

where $\mu > 0$ and $T > 0$ are model parameters and d_{ij} is the distance between points i and j in $\mathbb{H}^{d+1}$, given by Eq. (7). We refer to parameters T and μ as temperature and chemical potential, respectively, using the analogy with the Fermi-Dirac statistics. We note that the factors of 2 and ζ in Eq. (11) are to agree with the 2-dimensional RHG [2] that corresponds to $d = 1$.

Thus, RHG is formed in a two step network generation process:

1. Randomly select n points in $\mathbb{B}^{d+1}$ with pdf $\rho(\mathbf{x})$ in Eq. (11).

2. Connect i-j nodes pairs independently at random with distance-dependent connection probabilities $p_{ij} = p\left(d_{ij}\right)$, prescribed by Eq. (12).

Taken together, RHGs in $\mathbb{B}^{d+1}$ are fully defined by 6 parameters: properties of the hyperbolic ball, R_H and ζ; number of nodes n; radial component of node distribution α; chemical potential μ and temperature T.

Only four parameters, however, (n, α, T, R_H) are independent. It follows from (7) that ζ is merely a rescaling parameter for distances $\{d_{ij}\}$, and can be absorbed into r coordinates by the appropriate rescaling. Chemical potential μ controls the expected number of links and the sparsity of resulting network models. We demonstrate below that the sparsity requirement uniquely determines μ in terms of other RHGs parameters.

III. DEGREE DISTRIBUTION IN THE RHG

The structural properties of the RHG can be computed with the hidden variable formalism, Ref. [25], by treating node coordinates as hidden variables.

We begin by calculating the expected degree of node l located at point $\mathbf{x_l} = \{r_l, \theta_1^l, ..., \theta_d^l\}$:

$$\langle k(\mathbf{x_l})\rangle = (n-1)\int \frac{\mathrm{d}\mathbf{x_k}\rho(\mathbf{x_k})}{1 + e^{\frac{\zeta(d_{lk}-\mu)}{2T}}} \quad (13)$$

The symmetry in the angular distribution of points ensures that the expected degree of the node depends only on its radial coordinate r_l and not on its angular coordinates, $\langle k(\mathbf{x}_l)\rangle = \langle k(r_l, 0, .., 0)\rangle \equiv \langle k(r_l)\rangle$. This allows us to integrate out d angular coordinates in Eq. (13).

We also note that the choice of radial coordinate distribution given by Eq. (11) with $\alpha \geq \zeta/2$ results in most of the nodes having large radial coordinates, $r_i \approx R_H \gg 1$. This fact allows us to approximate distances using Eq. (9):

$$\langle k(r_l)\rangle = \int_0^{R_H}\int_0^{\pi} \frac{(n-1)\rho(r)\mathrm{d}r\rho_1(\theta)\mathrm{d}\theta}{1 + \left[e^{\zeta(r+r_l-\mu)}\left(\sin\left(\frac{\theta}{2}\right)\right)^2\right]^{\frac{1}{2T}}}. \quad (14)$$

Since the inner integral in Eq. (14) does not have a closed-form solution, to estimate $\langle k(r_l)\rangle$ we need to employ several approximations. We note that most nodes have large radial coordinates, $e^{(r+r_l-\mu)} \gg 1$, and the dominant contribution to the inner integral in (14) comes from small θ values. This allows us to estimate the integral by replacing $\sin(\theta)$ and $\sin(\theta/2)$ with the leading Taylor series terms, as $\sin(x) = x + \mathcal{O}\left(x^3\right)$:

$$\langle k(r_l)\rangle = \frac{1}{I_{d,1}}\int_0^{R_H}\int_0^{\pi} \frac{(n-1)\rho(r)\mathrm{d}r\theta^{d-1}\mathrm{d}\theta}{1 + \left[e^{\zeta(r+r_l-\mu)}\left(\frac{\theta}{2}\right)^2\right]^{\frac{1}{2T}}}, \quad (15)$$

where

$$I_{d,1} \equiv \int_0^{\pi} \sin^{d-1}(\theta)\mathrm{d}\theta = \sqrt{\pi}\Gamma\left[\frac{d}{2}\right]/\Gamma\left[\frac{d+1}{2}\right]. \quad (16)$$

The integral in Eq. (14) can be further simplified by the following change of RHG variables:

$$\begin{aligned} \{\mathfrak{r}, \mathfrak{r}_l, \mathcal{R}_H, \mathfrak{m}\} &= \frac{d\zeta}{2}\{r, r_l, R_H, \mu\}, \\ \tau &= dT, \\ \mathrm{a} &= 2\alpha/\zeta, \\ u &\equiv e^{(\mathfrak{r}+\mathfrak{r}_l-\mathfrak{m})}\left(\frac{\theta}{2}\right)^d. \end{aligned} \tag{17}$$

where the top line corresponds to 4 equations, each corresponding to one variable in the brackets.

In terms of the rescaled variables the connection probability function takes the form of

$$p_{ij} = \frac{1}{1+e^{\frac{\mathfrak{r}_i+\mathfrak{r}_j-\mathfrak{m}}{\tau}}\left[\sin\left(\frac{\Delta\Theta_{ij}}{2}\right)\right]^{\frac{d}{\tau}}}, \tag{18}$$

and Eq. (15) reads

$$\langle k(\mathfrak{r}_l)\rangle = \frac{2^d}{dI_{d,1}}\int_0^{\mathcal{R}_H}\int_0^{\left(\frac{\pi}{2}\right)^d}\frac{(n-1)\rho(\mathfrak{r})\mathrm{d}\mathfrak{r}\mathrm{d}\phi}{1+e^{\frac{\mathfrak{r}+\mathfrak{r}_l-\mathfrak{m}}{\tau}}\phi^{\frac{1}{\tau}}}, \tag{19}$$

where

$$\rho(\mathfrak{r}) \equiv \mathrm{a}e^{\mathrm{a}(\mathfrak{r}-\mathcal{R}_H)}. \tag{20}$$

By taking the inner integral in Eq. (19) we obtain

$$\begin{aligned} \langle k(\mathfrak{r}_l)\rangle &= \frac{(n-1)\pi^d}{dI_{d,1}}\int_0^{\mathcal{R}_H}\mathrm{d}\mathfrak{r}\rho(\mathfrak{r})\ {}_2F_1\left(1,\tau,1+\tau,-u_{max}^{\frac{1}{\tau}}\right), \\ u_{max} &= \left(\frac{\pi}{2}\right)^d e^{\mathfrak{r}_l+\mathfrak{r}-\mathfrak{m}}, \end{aligned} \tag{21}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

The expected average degree of the graph is given by

$$\langle k\rangle = \int_0^{\mathcal{R}_H}\mathrm{d}\mathfrak{r}\rho(\mathfrak{r})\langle k(\mathfrak{r})\rangle, \tag{22}$$

and degree distribution of the RHG can be expressed as

$$P(k) = \int_0^{\mathcal{R}_H}\mathrm{d}\mathfrak{r}\rho(\mathfrak{r})P(k|\mathfrak{r}), \tag{23}$$

where $P(k|\mathfrak{r})$ is a conditional probability that node with radial coordinate $\mathfrak{r}$ has exactly k connections.

In the case of sparse graphs $P(k|\mathfrak{r})$ is closely approximated by the Poisson distribution:

$$P(k|\mathfrak{r}) = \frac{1}{k!}e^{-\langle k(\mathfrak{r})\rangle}\left[\langle k(\mathfrak{r})\rangle\right]^k, \tag{24}$$

see Ref. [25], and the resulting degree distribution $P(k)$ is a mixed Poisson distribution:

$$P(k) = \frac{1}{k!}\int_0^{R_H}e^{-\langle k(\mathfrak{r})\rangle}\left[\langle k(\mathfrak{r})\rangle\right]^k\rho(\mathfrak{r})\mathrm{d}\mathfrak{r} \tag{25}$$

with mixing parameter $\langle k(\mathfrak{r})\rangle$.

IV. CONNECTIVITY REGIMES OF THE RHG

Depending on the value of the rescaled temperature $\tau = dT$, there exist three distinct regimes of the RHG: (i) cold regime ($\tau < 1$), (ii) critical regime ($\tau = 1$), and (iii) hot regime ($\tau > 1$). We provide detailed analyses these regimes below, and summarize our findings in Table I.

1. Cold regime, $\tau < 1$

In the $\tau < 1$ regime the hypergeometric function in (21) can be approximated as

$${}_2F_1\left(1,\tau,1+\tau,-u_{max}^{\frac{1}{\tau}}\right) = u_{max}^{-1}\frac{\pi\tau}{\sin(\pi\tau)}, \tag{26}$$

and $\langle k(\mathfrak{r}_l)\rangle$ and $\langle k\rangle$ are then given by:

$$\begin{aligned} \langle k\rangle &= (n-1)\frac{2^d}{dI_{d,1}}\frac{\pi\tau}{\sin(\pi\tau)}\langle e^{-\mathfrak{r}}\rangle^2 e^{\mathfrak{m}}, \\ \langle k(\mathfrak{r})\rangle &= \frac{\langle k\rangle}{\langle e^{-\mathfrak{r}}\rangle}e^{-\mathfrak{r}}, \end{aligned} \tag{27}$$

where $\langle e^{-\mathfrak{r}}\rangle \equiv \int_0^{\mathcal{R}_H}\mathrm{d}\mathfrak{r}\rho(\mathfrak{r})e^{-\mathfrak{r}}$. The explicit expression for $\langle e^{-\mathfrak{r}}\rangle$ follows from (20):

$$\langle e^{-\mathfrak{r}}\rangle = \frac{\mathrm{a}}{\mathrm{a}-1}\left(e^{-\mathcal{R}_H} - e^{-a\mathcal{R}_H}\right). \tag{28}$$

In the case $\mathrm{a} > 1$ we neglect the second term in (28) to obtain

$$\begin{aligned} \langle k(\mathfrak{r})(n)\rangle &\sim ne^{\mathfrak{m}-\mathfrak{r}_l-\mathcal{R}_H}, \\ \langle k(n)\rangle &\sim ne^{\mathfrak{m}-2\mathcal{R}_H}. \end{aligned} \tag{29}$$

Henceforth, we write $f(x) \sim g(x)$ when $\lim_{x\to\infty}\frac{f(x)}{g(x)} = K$, where $K > 0$ is a constant.

We note that $\langle k(\mathfrak{r})\rangle$ decreases exponentially as a function of $\mathfrak{r}$ with largest (smallest) expected degree corresponding to $\mathfrak{r} = 0$ ($\mathfrak{r} = \mathcal{R}_H$). By demanding that the largest and smallest expected degrees scale as

$$\langle k_{max}\rangle = \langle k(0)\rangle \sim n, \tag{30}$$

$$\langle k_{min}\rangle = \langle k(\mathcal{R}_H(n))\rangle \sim 1, \tag{31}$$

we obtain $\mathcal{R}_H \sim \ln n$ and $\mathfrak{m} = \mathcal{R}_H + \lambda$, where λ is an arbitrary constant.

First, we note that the scaling for $\mathcal{R}_H$ is consistent with our initial assumption of $\mathcal{R}_H \gg 1$ for large graphs. We also note that the exact value of the parameter λ is not important as long as it is independent of n. To be consistent with the original $\mathbb{H}^2$ formulation we set $\lambda = 0$, obtaining

$$\mathfrak{m} = \mathcal{R}_H = \ln(n/\nu), \tag{32}$$

where ν is the parameter controlling the expected degree of the RHG.

Applying scaling relationships (32) to (27) we obtain

$$\begin{aligned} \langle k\rangle &= \nu\frac{2^d}{dI_{d,1}}\left(\frac{a}{a-1}\right)^2\frac{\pi\tau}{\sin(\pi\tau)}\sim\mathcal{O}(1),\\ \langle k(\mathfrak{r})\rangle &= \frac{n}{\nu}\frac{a-1}{a}\langle k\rangle e^{-\mathfrak{r}}. \end{aligned} \tag{33}$$

Finally, using (24) and (25) we obtain

$$P(k) = (\gamma-1)\kappa_0^{\gamma-1}\frac{\Gamma[k-a]}{\Gamma[k+1]}\sim k^{-\gamma}, \tag{34}$$

where $\gamma = a+1$, and $\kappa_0 \equiv \frac{\gamma-2}{\gamma-1}\langle k\rangle$.

Hence, the cold regime corresponds to sparse scale-free graphs with $\gamma\in(2,\infty)$. We note that degree distribution is called scale-free if it takes the form of $P(k)=\ell(k)k^{-\gamma}$, where $\ell(k)$ is a slowly varying function, i.e., a function that varies slowly at infinity, see Ref. [26]. Any function converging to a constant is slowly varying. In the case of Eq. (34), $\ell(k)\to(\gamma-1)\kappa_0^{\gamma-1}$ as $k\to\infty$.

The special case of $a=1$ is also well defined. In this case the expressions for $\langle k\rangle$ and $\langle k(\mathfrak{r})\rangle$ given by Eq. (29) remain valid but $\langle e^{-\mathfrak{r}}\rangle$ is now given by

$$\langle e^{-\mathfrak{r}}\rangle = \mathcal{R}_H e^{-\mathcal{R}_H}. \tag{35}$$

As a result,

$$\begin{aligned} \langle k\rangle &= \nu\ln\left(\frac{n}{\nu}\right)^2\frac{2^d}{dI_{d,1}}\frac{\pi\tau}{\sin(\pi\tau)},\\ \langle k(\mathfrak{r})\rangle &= n\ln\left(\frac{n}{\nu}\right)\frac{2^d}{dI_{d,1}}\frac{\pi\tau}{\sin(\pi\tau)}e^{-\mathfrak{r}}. \end{aligned} \tag{36}$$

As seen from Eq. (36), $\langle k\rangle\sim(\ln(n))^2$, indicating that RHGs in this case are no longer sparse.

2. *Critical regime, $\tau=1$*

In the $\tau=1$, $a>1$ regime (21) and (22) up to the leading term can be approximated as:

$$\begin{aligned} \langle k\rangle &= (n-1)\frac{2^d}{dI_{d,1}}\left(\frac{a}{a-1}\right)^2\times\\ &\times e^{\mathfrak{m}-2\mathcal{R}_H}\ln\left[1+\left(\frac{\pi}{2}\right)^d e^{2\mathcal{R}_H-\mathfrak{m}}\right],\\ \langle k(\mathfrak{r})\rangle &= (n-1)\frac{2^d}{dI_{d,1}}\left(\frac{a}{a-1}\right)\times\\ &\times e^{\mathfrak{m}-\mathfrak{r}-\mathcal{R}_H}\ln\left[1+\left(\frac{\pi}{2}\right)^d e^{\mathfrak{r}+\mathcal{R}_H-\mathfrak{m}}\right], \end{aligned} \tag{37}$$

which in the case of scaling defined by (32) further simplify to

$$\begin{aligned} \langle k\rangle &= \frac{\nu 2^d}{dI_{d,1}}\left(\frac{a}{a-1}\right)^2\ln\left[1+\left(\frac{\pi}{2}\right)^d e^{\mathcal{R}_H}\right],\\ \langle k(\mathfrak{r})\rangle &= n\frac{2^d}{dI_{d,1}}\left(\frac{a}{a-1}\right)e^{-\mathfrak{r}}\ln\left[1+\left(\frac{\pi}{2}\right)^d e^{\mathfrak{r}}\right], \end{aligned} \tag{38}$$

indicating that in the critical regime, $\langle k(n)\rangle\sim\ln(n)$ and $\langle k(\mathfrak{r})(n)\rangle\sim n\mathfrak{r}e^{-\mathfrak{r}}$.

If $a=1$, (21) and (22) result in

$$\begin{aligned} \langle k\rangle &= \frac{\nu 2^d}{I_{d,1}}\ln\left(\frac{\pi}{2}\right)\mathcal{R}_H^2,\\ \langle k(\mathfrak{r})\rangle &= n\frac{2^d}{dI_{d,1}}e^{-\mathfrak{r}}\left[\mathcal{R}_H\left(\mathfrak{r}+d\ln\left(\frac{\pi}{2}\right)\right)-\frac{\mathcal{R}_H^2}{2}\right], \end{aligned} \tag{39}$$

indicating that $\langle k(n)\rangle\sim(\ln(n))^2$.

3. *Hot regime, $\tau>1$*

In the case $\tau>1$, (21) and (22) can be approximated as

$$\begin{aligned} \langle k\rangle &= (n-1)\left(\frac{\pi}{2}\right)^{1-\frac{1}{\tau}}\frac{2^d}{dI_{d,1}}\frac{\tau}{\tau-1}\left[\langle e^{-\mathfrak{r}/\tau}\rangle\right]^2 e^{\mathfrak{m}/\tau},\\ \langle k(\mathfrak{r})\rangle &= \frac{\langle k\rangle}{\langle e^{-\mathfrak{r}/\tau}\rangle}e^{-\mathfrak{r}/\tau}, \end{aligned} \tag{40}$$

where

$$\langle e^{-\mathfrak{r}/\tau}\rangle \equiv \int_0^{\mathcal{R}_H}\rho(\mathfrak{r})e^{-\mathfrak{r}/\tau}d\mathfrak{r}\approx\frac{a\tau}{a\tau-1}e^{-\mathcal{R}_H/\tau}. \tag{41}$$

Note that the expression for $\langle e^{-\mathfrak{r}/\tau}\rangle$ given by Eq. (41) is valid for all values of a and τ since $a\tau>1$.

Similar to the $\tau<1$ regime, we demand $\langle k_{max}(n)\rangle\sim n$ and $\langle k_{min}(n)\rangle\sim 1$ to obtain the scaling relationships for $\mathfrak{m}$ and $\mathcal{R}_H$:

$$\mathfrak{m}=\mathcal{R}_H=\tau\ln\left(n/\nu\right), \tag{42}$$

which in combination with (40) leads to

$$\begin{aligned} \langle k\rangle &= \frac{\nu 2^d}{dI_{d,1}}\left(\frac{\pi}{2}\right)^{1-\frac{1}{\tau}}\frac{a^2\tau^3}{(\tau-1)\left(a\tau-1\right)^2},\\ \langle k(\mathfrak{r})\rangle &= \frac{n}{\nu}\left(\frac{a\tau-1}{a\tau}\right)\langle k\rangle e^{-\mathfrak{r}/\tau},\\ P(k) &= a\tau\left[\langle k(\mathcal{R}_H)\rangle\right]^{a\tau}\frac{\Gamma[k-a\tau]}{\Gamma[k+1]}\sim k^{-\gamma}, \end{aligned} \tag{43}$$

where $\gamma=a\tau+1$.

V. LIMITING CASES OF THE RHG MODEL

In this section, we analyze several important parameter limits of the RHG and show that they correspond to well-defined graph ensembles.

A. $\tau\to 0$ limit in the cold regime

The case of $\tau=0$ is well-defined as the limit of $\tau\to 0$ of the cold regime. In this limit the connection probability

<table>
<tr><td>τ \ a</td><td>1</td><td>$(1, \infty)$</td><td>∞</td></tr>
<tr><td>0</td><td rowspan="3">$\mathfrak{m} = \mathcal{R}_H = \ln[n/\nu]$
$\langle k \rangle = O\left((\ln[n])^2\right)$
$\gamma = 2$</td><td rowspan="2">$\mathfrak{m} = \mathcal{R}_H = \ln[n/\nu]$
$\langle k \rangle = O(1)$
$\gamma = \mathrm{a} + 1$</td><td>Spherical Random Geometric Graph, $p_{ij} = \Theta(\theta_c - \Delta\theta)$</td></tr>
<tr><td>$(0, 1)$</td><td rowspan="3">Spherical Soft Random Geometric Graph, $p_{ij} = f(\Delta\theta)$</td></tr>
<tr><td>1</td><td>$\mathfrak{m} = \mathcal{R}_H = \ln[n/\nu]$
$\langle k \rangle = O(\ln[n])$
$\gamma = \mathrm{a} + 1$</td></tr>
<tr><td>$(1, \infty)$</td><td colspan="2">$\mathfrak{m} = \mathcal{R}_H = \tau \ln[n/\nu]$
$\langle k \rangle = O(1)$
$\gamma = a\tau + 1$</td></tr>
<tr><td>$\tau \to \infty$, $\lim_{\tau\to\infty} \zeta/\tau = \lambda$</td><td colspan="3">Hyper Soft Configurational Model, $\gamma = \frac{2\alpha}{d\lambda} + 1$</td></tr>
<tr><td>$\tau \to \infty$, $\lim_{\tau\to\infty} \mathfrak{m}/\tau = \lambda$</td><td colspan="3">$G(n, p)$, $p = \frac{1}{1+e^{-\lambda}}$</td></tr>
</table>

<table>
<tr><td>T \ α</td><td>$\frac{\zeta}{2}$</td><td>$\left(\frac{\zeta}{2}, \infty\right)$</td><td>$\infty$</td></tr>
<tr><td>0</td><td rowspan="3">$\mu = R_H$, $R_H \sim \ln[n]$
$\langle k \rangle = O\left((\ln[n])^2\right)$
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<tr><td>$\frac{1}{d}$</td><td>$\mu = R_H$, $R_H \sim \ln[n]$
$\langle k \rangle = O(\ln[n])$
$\gamma = 2\frac{\alpha}{\zeta} + 1$</td></tr>
<tr><td>$\left(\frac{1}{d}, \infty\right)$</td><td colspan="2">$\mu = R_H$, $R_H \sim T\ln[n]$
$\langle k \rangle = O(1)$
$\gamma = 2\frac{\alpha}{\zeta} dT + 1$</td></tr>
<tr><td>$T \to \infty$, $\lim_{T\to\infty} \frac{\zeta}{T} = \mathrm{d}\lambda$</td><td colspan="3">Hyper Soft Configurational Model $\gamma = \frac{2\alpha}{d\lambda} + 1$</td></tr>
<tr><td>$T \to \infty$, $\lim_{T\to\infty} \frac{\zeta\mu}{2T} = \lambda$</td><td colspan="3">$G(n, p)$, $p = \frac{1}{1+e^{-\lambda}}$</td></tr>
</table>

TABLE I: RHG regimes in (top) rescaled and (bottom) original variables.

function in Eg. (12) becomes the step function, and connections are established deterministically between node pairs separated by distances smaller than μ.

In this case $\frac{\pi\tau}{\sin(\pi\tau)} \to 1$ leading to

$$\langle k\rangle = \nu\frac{2^d}{dI_{d,1}}\left(\frac{\mathfrak{a}}{\mathfrak{a}-1}\right)^2, \quad (44)$$

$$\langle k(\mathfrak{r})\rangle = \frac{n}{\nu}\left(\frac{\mathfrak{a}-1}{\mathfrak{a}}\right)\langle k\rangle e^{-\mathfrak{r}}. \quad (45)$$

The resulting graphs are sparse and are characterized by the scale-free degree distribution $P(k) \sim k^{-\gamma}$, $\gamma = \mathfrak{a}+1$.

B. $\tau \to \infty$ limit: Erdős Rényi model

The case of $\tau \to \infty$, on the other hand, is not the special case of the hot regime. Indeed, it follows from Eq. (42) that $e^{\frac{\mathfrak{r}+\mathfrak{r}_l-\mathfrak{m}}{\tau}} \sim \frac{1}{n} \ll 1$ and the small angle approximation of $\sin(x) \approx x$ in Eq. (15) no longer holds.

Therefore, to infer the proper RHG behavior in the $\tau \to \infty$ limit, we retreat back to Eq. (12) and set $T \to \infty$.

We observe that in this case the RHG degenerates to the Erdős Rényi (ER) model with connection probability

$$p_{ij} \to p = \frac{1}{1+e^{-\lambda}}, \quad (46)$$

where

$$\lambda \equiv \lim_{T\to\infty} \zeta\mu/2T = \mathfrak{m}/\tau, \quad (47)$$

Degree distribution of the HRG in this limit is binomial

$$P(k) = \binom{n-1}{k} p^k \left[1-p\right]^{n-k-1}, \quad (48)$$

and

$$\langle k\rangle = (n-1)p = \frac{n-1}{1+e^{-\lambda}}, \quad (49)$$

Since binomial distribution at large k values decays faster than any power-law function $k^{-\gamma}$ with finite positive γ we refer to this regime as $\gamma = \infty$ case in Table I.

C. $\mathfrak{a} \to \infty$ limit: Soft Random Geometric Graphs

In this limit radial node distribution degenerates to

$$\rho(\mathfrak{r}) \to \delta\left(\mathfrak{r} - \mathcal{R}_H\right). \quad (50)$$

As a result, all nodes are placed at the boundary of the hyperbolic ball $\mathbb{B}^{d+1}$ with $\mathfrak{r}_i = \mathcal{R}_H$. Even though the distances between nodes are still hyperbolic, they are fully determined by the angles on $\mathbb{S}^d$:

$$\zeta d_{ij} = \cosh^{-1}\left[\cosh\left(\frac{2\mathcal{R}_H}{d}\right)^2 - \sinh\left(\frac{2\mathcal{R}_H}{d}\right)^2 \sin\left(\Delta\theta_{ij}\right)\right].$$

Hence, connection probabilities $\{p_{ij}\}$ are fully determined by angles $\Delta\theta_{ij}$:

$$p_{ij} = p_{ij}\left(\Delta\theta_{ij}\right) = \frac{1}{1+\exp\left(\frac{\tilde{d}_{ij}(\Delta\theta_{ij})-\mathfrak{m}}{\tau}\right)}, \quad (51)$$

where $\tilde{d}_{ij}\left(\Delta\theta_{ij}\right) \equiv \frac{d\zeta}{2} d_{ij}\left(\Delta\theta_{ij}\right)$.

Effectively, in the $\mathfrak{a} \to \infty$ regime nodes are placed at the surface of the unit sphere $\mathbb{S}^d$ and connections are made with distance-dependent probabilities on the sphere. Hence, RHG in the $\mathfrak{a} \to \infty$ limit can be viewed as the soft RGG on $\mathbb{S}^d$.

D. $\mathfrak{a} \to \infty$, $\tau \to 0$ limit: Random Geometric Graphs

If $\mathfrak{a} \to \infty$ and $\tau \to 0$ connection probabilities in Eq. (12) become

$$p_{ij} = \Theta(\theta_c - \Delta\theta_{ij}), \quad (52)$$

where θ_c is the solution to the equation $\tilde{d}_{ij}(\theta_c) = \mathfrak{m}$. Thus, the RHG in the $T \to 0$ limit becomes the sharp random geometric graph (RGG) on $\mathbb{S}^d$.

Since nodes are uniformly placed on $\mathbb{S}^d$, resulting degree distribution of the RGG is binomial

$$P(k) = \binom{n-1}{k} \tilde{p}^k \left[1-\tilde{p}\right]^{n-k-1}. \quad (53)$$

Here $\tilde{p}$ is the probability for a randomly chosen node to end up within the angle of θ_c of the $\theta_1, ..., \theta_d$ point:

$$\tilde{p} = \frac{\int_0^{\theta_c} \left[\sin\left(\theta\right)\right]^{k-1} \mathrm{d}\theta}{\int_0^{\pi} \left[\sin\left(\theta\right)\right]^{k-1} \mathrm{d}\theta}. \quad (54)$$

Since binomial distribution at large k values decays faster than any power-law function $k^{-\gamma}$ with finite positive γ we refer to this regime as $\gamma = \infty$ case in Table I.

E. $\zeta \to \infty$, $\tau \to \infty$ limit: Hyper Soft Configuration Model

In the $\zeta \to \infty$ limit hyperbolic distances degenerate to

$$d_{ij} = r_i + r_j. \quad (55)$$

Further, if $\lim_{\zeta\to\infty} \frac{\zeta}{\tau} = \lambda > 0$, where λ is a constant, connection probability in Eq. (12) simplifies to

$$p_{ij} = \frac{1}{1+e^{\omega_i}e^{\omega_j}}, \quad (56)$$

resulting in the Hyper Soft Configuration Model (HSCM). Here $\omega_i = \frac{d\lambda}{2}\left(r_i - \frac{\mu}{2}\right)$ are Lagrange multipliers controlling expected node degrees. Lagrange multipliers are drawn from effective pdf

$$\rho(\omega) = \frac{2\alpha}{d\lambda} e^{\alpha\left(\frac{\mu}{2} - R_H\right)} e^{\frac{2\alpha}{d\lambda}\omega}, \quad (57)$$

$$\omega \in \left(-\frac{d\lambda\mu}{4}, \frac{d\lambda}{2}\left(R - \frac{\mu}{2}\right)\right). \quad (58)$$

Expected degrees in the HSCM are approximated by

$$\begin{aligned} \langle k(\omega_i)\rangle &= (n-1)\langle e^{-\omega}\rangle e^{-\omega_i}, \\ \langle k\rangle &= (n-1)\langle e^{-\omega}\rangle^2 \end{aligned} \quad (59)$$

By demanding that $\langle k\left(\omega(r=0)\right)\rangle \sim n$ and $\langle k\left(\omega(r=R_H)\right)\rangle \sim 1$ we obtain $R_H = \frac{2}{d\lambda}\ln(n)$, while $\mu = R_H$ in the case of $\frac{2\alpha}{d\lambda} > 1$, and $\mu = \frac{2\alpha}{d\lambda}R_H$ in the case of $\frac{2\alpha}{d\lambda} < 1$.

In both cases $\langle k(\omega)\rangle \sim e^{-\omega}$ and graphs are characterized by the scale-free degree distribution:

$$\begin{aligned} P(k) &\sim k^{-\gamma}, \\ \gamma &= \frac{2\alpha}{d\lambda} + 1. \end{aligned} \quad (60)$$

VI. SUMMARY

In our work we have generalized the RHG to arbitrary dimensionality. In doing so, we have found the rescaling of network parameters, given by Eq. (17), that allows to reduce RHGs of arbitrary dimensionality to a single mathematical framework. Summarized in Table I, our results indicate that RHGs exhibit similar topological properties, regardless of the dimensionality of their latent hyperbolic spaces.

While the dimensionality of the hyperbolic space does not seem to significantly degree distributions of the RHG models, we conjecture that higher-dimensional RHGs may be instrumental in mapping of real networks. One of the standard mapping approaches is the Maximum Likelihood Estimation (MLE), finding node coordinates of the network of interest by maximizing the likelihood that the network was generated as the RHG in the latent space. The likelihood function in the case of the $\mathbb{B}^2$ has been shown to be extremely non-convex with respect to node coordinates [15], making the standard learning tools like the stochastic gradient descent inefficient. Raising the dimensionality of the $\mathbb{H}^{d+1}$ may lift some of the local maxima of the likelihood function, resulting in faster and more accurate network mappings.

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